

PAPER • OPEN ACCESS

## Mental construction in mathematical proof

To cite this article: I W E Mahendra *et al* 2019 *J. Phys.: Conf. Ser.* **1321** 022135

View the [article online](#) for updates and enhancements.



**IOP | ebooks™**

Bringing you innovative digital publishing with leading voices to create your essential collection of books in STEM research.

Start exploring the collection - download the first chapter of every title for free.

# Mental construction in mathematical proof

I W E Mahendra\*, N K Erawati and N W Sunita

Education Mathematics Department, Mathematics and Science Education Faculty,  
IKIP PGRI BALI, Indonesia

\*Corresponding author: [eka\\_undiksha@yahoo.com](mailto:eka_undiksha@yahoo.com)

**Abstract.** Mental construction in this study is mental construction based on APOS theory (action, process, object, schema) in conducting mathematical proof. Therefore the purpose of this research is to find out how mental construction is based on APOS theory in mathematical proof and the results obtained from its application. This study included a quasi-experimental with posstest only nonequivalent comparison group design. This study conducted at IKIP PGRI Bali. The research sample was students who took real analysis courses. Based on the sampling results obtained two classes. Data retrieval is done by direct observation and tests. Based on the observations, it shows that the student worksheet is able to make all students active in learning. Data analysis shows that the mental construction of students proving ability is only up to the object stage. Based on the whole test results, the experimental class achieved significantly better results compared to the comparison class. This shows that the mental construction of students in proof has not yet reached the schema stage, which means that it needs to be improved and the application of APOS learning is able to help students in mathematical proof.

## 1. Introduction

Learning mathematics, especially at the university level, will not be separated from mathematical proof activities. Proof is a series of logical arguments that explain the truth of a statement. The ability to proof in mathematics consists of the ability to construct proof and the ability to validate proof. The ability to construct proof includes the ability to use proof methods, definitions, entries, and theorems to show the truth of a statement in mathematics. While the ability to validate proof includes the ability to criticize proof related to the types of proof that often appear in mathematics.

The difficulty in learning mathematical proof is application of mathematical language, understanding the meaning of the proposition, knowing where to get started on a proof, using methods to construct the proof, and defining logical structures of the proposition to construct the proof [1]. The other research find seven difficulties experienced by students in conducting proof there are students do not know the definition and they cannot state the definition, students have a little intuitive understanding of the concept, student concept drawings are not sufficient to prove, students not able to build and using their own examples, students do not know how to use definitions to determine the whole structure of proof, students unable to understand and use language and mathematical notation, students do not know how to start proof[2]. Difficulties in doing proof also occurred to students of mathematics education at IKIP PGRI Bali. Based on the results of interviews with several lecturers of



mathematics courses revealed that students were still weak in conducting proof. The most basic obstacle experienced is that students do not understand the statement about what must be proven so they do not know the initial steps in showing proof. Therefore, it is necessary to develop a method and tool with a clear framework so that it can guide students in mathematical proof.

One learning theory that has an appropriate framework in mathematical proof activities is the mental construction of APOS (action, process, object, schema). APOS is a framework used to explain how a person constructs his understanding of a mathematical concept [3]. APOS theory assumes that a individual mathematical knowledge is the result of interaction with others and the results of the person's mental construction in understanding mathematical ideas [3]. Mathematical proof can develop high thinking skills (Hots). Research on Hots with problem-based learning combined with the computer-assisted APOS theory shows better problem-solving abilities compared to classes that were not treated with the APOS theory [4]. But the use of computers can provide obstacles for students when the facilities provided by the college do not support. Therefore, the use of student worksheet becomes a solution that can be done even if the student or college has inadequate facilities.

The use of student worksheet can help students determine the flow of thinking in compiling a mathematical proof because the student worksheet is designed according to APOS mental construction. Therefore, it can be determined the purpose of the research is to explain how mental construction is based on APOS theory in mathematical proof and the results obtained from its application.

## 2. Methods

This research is quasi-experimental with posttest only nonequivalent comparison group design using one experimental class and one comparison class. This design is used because experiments conducted cannot comparison the sample from other factors and the things observed are differences in the results obtained due to the treatment given. Samples were chosen with purposive random sampling and then simple random sampling. The first sampling give three class, and the second sampling give two class. The research sample is VIA class is obtained as an experimental class and VIB is a comparison class. This study conducted at IKIP PGRI Bali at March until July 2017, especially for students who take real analysis courses. The reason for choosing the real analysis course is because this course contains many abstract theorems and concepts so that students need to master the basic concepts that can be used in mathematical proving. In addition, students at this level are considered to have a strong foundation because they have been in third years.

The stages carried out in this study consisted of three stages: preparation, implementation and finishing of research. Prepare of students worksheet, instruments, and validation of instrument was done at preparation stage. The research was carried out by giving treatment to the APOS class and conventional learning in the comparison class. At the end of the study both classes were given a test. data were analyzed by descriptive statistics and t test.

## 3. Results and Discussion

Based on the results of a review of the APOS theory, APOS mental construction framework can be arranged in the mathematical proof shown in Table 1.

**Table 1.** APOS Mental Construction Framework in Proof of Mathematics

APOS Mental Construction	APOS Mental Construction Framework in Proof
Action	Students carry out activities in small groups by answering questions from the student's worksheet. At this stage presented questions that are arranged in a coherent way to help students understand concepts and definitions and students display each answer explicitly. Definition becomes the basis for proof. For example, the questions are made in order to get to the definition of series. This will support

Process	the emergence of mental construction of action. At the process stage, students are confronted again with the question as a repetition of more profound action stage. At this stage it is possible to reflect on what has been done in the action stage and mental construction shifts to the process stage. Students can use the right mathematical language to express ideas and arrange steps (actions) without explicitly showing each step. For example, students do proof based on definitions that have been understood at the action stage.
Object	Students are able to take action and provide reasons or further explanation. The link in proof is that students are able to explain each step in the proof and provide the right reasons. The appearance of the object stage is done with the presentation and discussion in or between groups about what has been understood in the action and process stages. Discussions and presentations can make what has been understood settle deeply in the minds of students.
Schema	The student can fully understands and can use a concept that has been understood in solving the problems. All processes that have taken place at the stage of action, processes and objects are connected and organized together in the minds of individuals. The schema phase is facilitated by the practice of questions at the end of the discussion without assistance from the lecturer.

The APOS mental construction framework that has been arranged is applied through the students worksheet given to the experimental class. Furthermore, learning is carried out in the form of an students worksheet and the use of other learning resources that can help students answer questions in the students worksheet. Whereas in the comparison class that occurs is learning that is done by the presentation method and discussion of student learning outcomes about a material without the existence of an students worksheet. At the end of the study, students were given a test to measure the ability to prove which involved all the material discussed in the students worksheet. The test results provide data summarized in Table 2.

**Table 2.** Scores of Students' Mathematical Proof Ability

	Group	
	Experiment	Comparison
Average score	53.78	41.09
Maximum score	75	60
Minimum score	30	25

Based on the data in Table 2 shows that the average score of the proof ability the experimental group is higher than the comparison group. Furthermore based on the set value criteria, the ability of students is only up to GOOD criteria and only achieved in the experimental class.

The next step is to conduct a prerequisite test, which is normality test and data homogeneity test. Normality test uses Kolmogorov-Smirnov with SPSS for Windows 16.0. Based on the results of the analysis it can be seen that the data is normally distributed, then the homogeneity test with Levene's Test with SPSS for windows 16.0 show the same variance between groups. Because the variance between groups is the same, it can be concluded that the group being tested is homogeneous. Prerequisite tests have been fulfilled, then hypothesis testing is done with t test. Based on the test results obtained t count of 4.546 with sig values equal to 0,000 so that  $H_0$  is rejected, which means that the mathematical proof ability between the experimental and comparison groups is significantly different.

The APOS mental construction framework that has been prepared has given the opportunity for each stage of mental construction from APOS theory. Furthermore, the framework is outlined in an students worksheet that contains questions that can generate the construction of knowledge in the

individuals minds of both expressed explicitly and implicitly. Steps are done explicitly by students indicate that students have reached the "action" stage. Students worksheet are carried out in small groups of three to four people so that learning and interaction occur optimally. Interaction is needed to generate the occurrence of processes and objects from APOS. Interaction in groups allows the exchange of ideas from each student to solve the problem so that it raises the "process" stage. Intergroup interaction occurs when each group presents its work by giving a more detailed explanation so that there is discussion in the class that is able to bring up the "object" stage. In order to have a good and smooth discussion, the lecturer plays a role in supervising and directing students to achieve the learning objectives in accordance with the time provided, and the lecturer plays a role in correcting the concept errors that occur directly. Furthermore, the discussion ended with a joint question exercise on the material that was discussed.

Proof in mathematics is not only important for students but also for teachers. This is supported by research conducted by Lesseig who designed the framework for teachers and students in learning proof. The existence of a clear framework can help mathematical proof learning activities [5]. The application of the APOS mental construction framework provides good results. The results showed that the mental construction of the APOS theory was able to help students in studying the isomorphism theorem in abstract algebra<sup>3</sup>. Based on the results of data analysis, it has been shown that the proof ability achieved by APOS mental construction learning is better than conventional learning. This happens because students get the opportunity to learn gradually by answering questions that are arranged in a coherent and learning situation that allows students to think and construct their own knowledge. The results of the construction in mind are immediately expressed in writing and verbally through discussion. Furthermore, if there is a construction error regarding a concept, the error can be immediately known and corrected so that students can obtain the correct understanding. In addition, mental construction raised with the activity of studying the existing proof and clarifying the proving step with the sentence itself. Through APOS mental construction that is done repeatedly, students are used to learning to understand and write down proof, not just memorizing proof. The application of APOS theory in learning was designed by genetic decomposition also conducted by Salgado who concluded that the genetic decomposition applied was promising in learning eigenvalues and eigenvectors [6]. APOS study also conducted by Lorca which reported several relevant findings on mental construction related to groups, normal subgroups, and subgroups of results obtained using APOS theory [7].

The learning occurs in the comparison class is the lecturer delivering the material lecturely and students are assigned to present the material that has been determined. The presentation carried out contains proof obtained from books or other sources, but does not understand each step of the proof. This method makes students unwilling to ask more details about what their friends presented because they did not understand the material described. Furthermore, if there are no students asking, then the learning continues in the next material. This resulted in a lack of understanding of students about a mathematical proof.

Beside the difference of significance of the proof ability, between the experimental and comparison classes, the observations show that students studied in the APOS framework are more active in carrying out tasks, asking questions and participating in class discussions. Whereas in the comparison class, namely the presentation method, students who are not on duty during the presentation only become good listeners, even though they do not understand what their friends are presenting.

APOS framework to support the occurrence of objects must be done with social interaction both in small groups and in the whole class. This interaction is what brings communication and trains students psychologically in social interaction in their class. The interaction created in the APOS mental construction provoked and forced each student to work in his group and try to understand the material being studied. Every student has the opportunity to express opinions and is required to express his opinion. Dissenting opinions will certainly further spur an interesting discussion. This method is very effective for students in learning material that contains abstract concepts. The discussion activity

which ends with the practice of questions is the first step for students to show that what they did before was truly understood.

Based on student answers, it can also be known the mental construction stages achieved by students. The indicators that can be used are as follows. (1) The answers that reflect the achievement of the "action" stage are when students are able to identify what is known and what will be proven which is seen from what is written. (2) Next entering the process stage students will be able to write proof based on definitions or appropriate theorems. (3) The stage of the object is seen from when the student is able to write an explanation regarding the relation of the definition or theorem used in the problem being solved or provide a more detailed explanation of the main steps in the proof made. (4) The final stage is the schema stage can be seen from the complete truth of the three previous stages and write down the final conclusions obtained, or resolve the problem thoroughly.

Based on these indicators it can be seen that overall of all the material tested there were no students who reached the schema phase perfectly. The perfect schema in question is that students are able to reach the schema of the four test questions tested. This can be seen from the results achieved by students that are only able to be at Good criteria (B). This is in the test questions that contain trigonometric derivatives. Lack of students' knowledge with trigonometric rules makes it difficult for students to complete the required proof. Whereas in the last material that is the part of the metric space combined with the rules of absolute value makes students unable to complete the proof, even though students already know the definitions that must be met in the metric space but students forget the rules in absolute values, so that the proof cannot be done correctly. Then the object stage is achieved by 40% of students. This amount is seen from the number of students who get a score above 60, because by achieving a minimum score of 60 means that students have passed the process stage (stage 50%) but have not reached the object stage perfectly. Furthermore, a number of 60% of students were able to reach the process stage which is seen from the minimum final score of 50. The most basic stage is the action stage that can be achieved by 80% of students; at this stage, students reach a minimum score of 40. The remaining 20% of students have not been able to reach the action stage is perfect because it does not understand the mathematical statement that must be proven. Syamsuri's research results show the students were unable to construct a formal proof because have no proof-structure, and also have a little of conceptual understanding [8]

Empirical studies also illustrate mathematics majors' difficulties with reading proofs by demonstrating that students often cannot distinguish between valid and invalid arguments [9]. Hence requires a framework that can help students start writing proof in mathematics. Selden and Selden says in their research, since every proof can be constructed using a proof framework, we consider constructing proof frameworks as a reasonable place to start. Also, helping students interpret formal mathematical definitions so that these become operable might be another place to start. This would be helpful because one often needs to convert a definition into an operable form in order to use it to construct a second-level framework [10]. The analysis of students' answers was also carried out by Maharaj. The findings of this study indicate that students experience difficulties in applying rules to derivatives and many students do not have the right mental structure in processes, objects and schema levels. This study shows that a strong foundation in understanding basic knowledge is very important before learning higher level knowledge [11].

#### 4. Conclusions

Based on the results and discussion that has been explained, it can be concluded that APOS mental construction is effective in activating students in class and helping students in mathematical proof. APOS mental construction stages achieved by students have not reached the schema stage perfectly and there are students who have not yet reached the action stage. Therefore, The presentation of the stages achieved by students shows that the ability to prove mathematics must be improved by creating a learning atmosphere that supports and trains students' abilities from the beginning so that at the final stage the ability of students is able to reach the schema stage which is able to organize their knowledge in full and solid in mathematical proof.

### Acknowledgments

Thank you to Kemenristekdikti for the funds provided through DIPA fund grants and IKIP PGRI Bali for the opportunities that have been provided as institutions.

### References

- [1] Uğürel I, Moralı H S, Koyunkaya M Y and Karahan O 2016 *EURASIA J. Math. Sci Tech.* **12** 203
- [2] Maya R and Utari S 2011 *IndoMS JME* **2** 231
- [3] Arnon I, Cottrill J, Dubinsky E, Oktac A, Trigueros S R F M and Weller K 2014 *APOS Theory A Framework for Research and Curriculum Development in Mathematics Education* (New York: Springer)
- [4] Mudrikah A 2016 *IJRES* **2** 125
- [5] Lesseig K 2016 *IJRES* **2** 253
- [6] Salgado H and Trigueros M 2015 *ELSEVIER J.Math.Behav.* **39** 100
- [7] Lorca A M and Parraguez A M M 2016 *IEJME Math. Educ.* **11** 377
- [8] Syamsuri, Purwanto, Subanji, Irawati S 2017 *IJEME* **1** 135
- [9] Weber K 2015 *Int. J. Res. Undergrad. Math. Ed.* **1** 289
- [10] Selden A and Selden J 2015 *CERME 9<sup>th</sup> Congress of the European Society for Research in Mathematics Education*, Charles University in Prague (Faculty of Education ERME Feb 2015 Prague, Czech Republic) pp 198-204
- [11] Maharaj A 2013 *S. Afr. J. Educ.* **33** 1